COM-202 - Signal Processing

Homework 10

Please submit your answer to Exercise 1 by May 8, 2025

Exercise 1. Continuous-time Fourier Transform

(a) Using the Fourier transform formula, find the Fourier transform of the following signals

- $x_1(t) = e^{-at}u(t)$, with $Re(a) > 0$
- $x_2(t) = e^{at}u(-t)$, with $Re(a) > 0$

Recall that the unit step $u(t)$ is defined as

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0. \end{cases}$$

(b) Using the Fourier transform formula, prove the following properties of the continuous-time Fourier transform

- Scaling property: $x(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{f}{a}\right)$ where $a \neq 0$
- Shift in time property: $x(t - t_0) \overset{\text{CTFT}}{\longleftrightarrow} e^{-j2\pi f t_0}X(f)$

Exercise 2. Sampling Sinusoids

(a) Consider a sampler operating at a sampling frequency $F_s = 500$ Hz. Which of the following signals can be converted to discrete-time sequences with no loss of information by this system?

- $x_1(t) = \cos(2\pi f_1 t)$, with $f_1 = 100$ Hz
- $x_2(t) = \sin(2\pi f_2 t)$, with $f_2 = 225$ Hz
- $x_3(t) = \sin(2\pi f_3 t)$, with $f_3 = 1250$ Hz
- $x_4(t) = \cos(2\pi f_1 t) + \sin(2\pi f_4 t)$, with $f_1 = 100$ Hz and $f_4 = 400$ Hz

(b) A second sampler operates by sampling its input every $T_s = 0.5\times 10^{-3}$ seconds. Which of the following signals can be converted to discrete-time sequences with no loss of information by this system?

- $x_5(t) = \cos(2\pi f_5 t)$, with $f_5 = 500$ Hz
- $x_6(t) = \sin(2\pi f_3 t)$, with $f_3 = 1250$ Hz
- $x_7(t) = \cos(2\pi f_6 t) + \sin(2\pi f_7 t)$, with $f_6 = 250$ Hz and $f_7 = 150$ Hz
- $x_8(t) = \sin(2\pi f_8 t)$, with $f_8 = 750$ Hz

Exercise 3. Raw sampling

The continuous-time signal

$$x(t) = \sum_{m=1}^{4} m\cos(2\pi f_0 m\, t)$$

with $f_0 = 300$ Hz, is raw-sampled into the discrete-time signal $x[n] = x(nT_s)$ using $T_s = 5\cdot 10^{-4}$ seconds. Sketch the DTFT of $x[n]$.

Exercise 4. Bandwidth of a signal

Consider a bandlimited continuous-time signal $x(t)$ whose total bandwidth is W Hz (in other words, the spectrum $X(f)$ is zero for $|f| > W/2$. Determine the maximum possible bandwidth for each of the following signals, assuming that $X(f) \neq 0$ over its entire bandwidth:

(a) $x_1(t) = x(t) - x(t-1)$

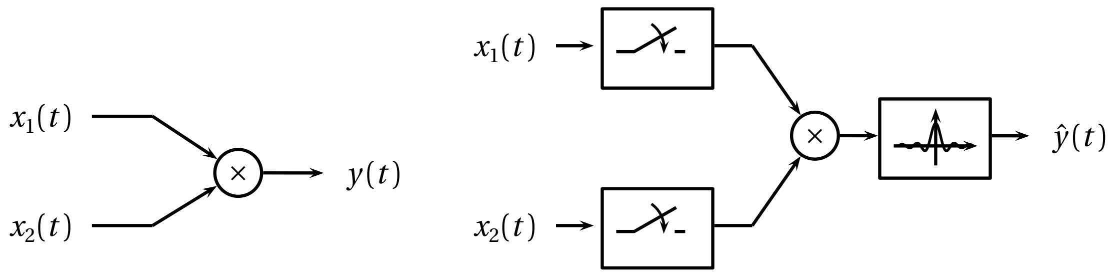
(b) $x_2(t) = x^2(t)$

(c) $x_3(t) = 2x(t)\cos(2\pi W t)$

(d) $x_4(t) = (x * h)(t)$ where $h(t) = \mathrm{sinc}((W/3)t)$

Exercise 5. Discrete-time implementation of analog systems

Consider the continuous-time system shown in figure (a) below, whose output is the product of its two input signals. In order to implement a discrete-time version of this system, you build the device shown in figure (b), using two samplers and an ideal sinc interpolator, all of which work at the same rate F_s.

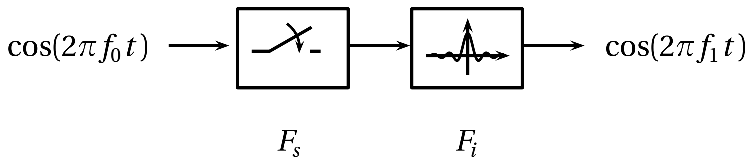


(a) (b)

You know that the real-valued, continous-time input signals are bandlimited, with a maximum positive frequency $F_N = 8000$ Hz. Determine the minimum value for the rate F_s so that the discrete-time implementation produces exactly the same output as the continuous-time original system. Explain in detail your choice and, if in doubt, "test" the discrete-time system using the input signals $x_1(t) = x_2(t) = x(t) = \text{sinc}(2F_N t)$.

Exercise 6. Mystery Signal

Consider the following setup, where a sinusoidal input of unknown frequency is first raw-sampled at a rate $F_s = 500$ Hz and then sinc-interpolated at a rate $F_i = 250$ Hz.



You measure the frequency of the output sinusoid and find out that $f_1 = 50$ Hz. Which of the following input frequencies would produce the measured output?

(a) $f_0 = 100$ Hz

(b) $f_0 = 150$ Hz

(c) $f_0 = 400$ Hz

(d) $f_0 = 600$ Hz

Exercise 7. Aliasing in Time?

Consider an N-periodic discrete-time signal $\tilde{\mathbf{x}}$, with N an *even* integer, and let $\tilde{\mathbf{X}}$ be its N-point DFS:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}nk} \qquad k \in \mathbb{Z}$$

Consider now a vector $\tilde{\mathbf{Y}}$ of length $N/2$ obtained by selecting the even-numbered elements of $\tilde{\mathbf{X}}$:

$$Y[m] = \tilde{X}[2m], \quad m = 0, 1, \ldots, N/2.$$

If we now compute the inverse DFS of $\tilde{\mathbf{Y}}$ we obtain the $(N/2)$-periodic signal $\tilde{\mathbf{y}}$

$$\tilde{y}[n] = \frac{2}{N} \sum_{k=0}^{N/2-1} \tilde{Y}[k] e^{j \frac{2\pi}{N/2} nk} \qquad n \in \mathbb{Z}.$$

Express $\tilde{\mathbf{y}}$ in terms of $\tilde{\mathbf{x}}$ and describe their relationship.